David C. Folta

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TECHNICAL REPORT

MATHEMATICAL FUNDAMENTALS OF TRAJECTORY DYNAMICS

by

R. Donati

B. B. Gragg

G. S. Kondos

Lockheed

MISSILES & SPACE COMPANY

A GROUP DIVISION OF LOCKHEED AIRCRAFT CORPORATION

SUNNYVALE, CALIFORNIA

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FOREWORD

The purpose of this report is to present those fundamentals of classical analytic mechanics which are necessary for understanding the simulation of space vehicle motion and performance. Emphasis is placed on concepts, principles, and methods which are especially suitable for use in developing mathematical models of missile trajectory problems for solution on digital computers.

For training purposes, there is presented a unified, yet detailed, discussion of coordinate geometry (especially translations and rotations), elementary matrix and vector operations, and the classical Newtonian physics governing the dynamics of particles and rigid bodies. Such an integrated approach, which is required for familiarization of junior engineers and computer programmers with trajectory simulation, is not generally found in the usual textbooks on physics, calculus, and analytic geometry. Furthermore, this manual provides complete development of certain vital theorems which are either omitted in the textbooks or treated as obscure exercises. All topics are developed with mathematical rigor so that the document can serve as a source book on fundamental questions and principles employed in LMSC digital computer trajectory programs.

This report can serve as an instructional manual for people who have studied analytic geometry and elementary calculus but no classical dynamics. It also can be used for review study by individuals who have taken academic courses in mechanics.

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Section 1
EULER ANGLE COORDINATE TRANSFORMATIONS

1.1 INTRODUCTION

Describing the full dynamics of a missile trajectory involves the use of a number of coordinate reference frames in which the various physical relationships are defined mathematically as vector components. This requires a means of specifying the relative orientation of one frame of reference with respect to another and a systematic method of performing coordinate transformations.

Expressing relationships in matrix and vector notation not only facilitates transformations, but preserves much physical intuition embodied in the notation. The use of coordinate transformations makes it possible and desirable to define physical quantities in their most convenient frames of reference.

This section first presents the basic matrix and vector operations required to understand coordinate transformations, then describes the use of Eulerian angles and direction cosines in accomplishing these transformations.

1.2 ELEMENTARY MATRIX OPERATIONS

1.2.1 A matrix A is a rectangular array of elements of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If $m = n$, then matrix A is called a square matrix.

The order of a matrix is indicated by the number of rows and columns that form its array. For example, matrix A is of order m by n , denoted $m \times n$, where m is the number of rows and n the number of columns. If $m = n$, the matrix is said to be of order m .

1.2.2 A convenient way of denoting matrix A is (a_{ij})

where

$$i = 1, 2, \ldots, m$$
$$j = 1, 2, \ldots, n$$

and a_{ij} indicates the element in the i^{th} row and j^{th} column.

1.2.3 The transpose A^T of matrix A is

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

which is obtained from A by interchanging rows and columns.

1.2.4 A matrix $\bar{x}$, consisting of one row and n columns, is called an nth order row vector.

$$\bar{x} = (x_1 , x_2 , \ldots , x_n)$$

1.2.5 The transpose, $\bar{x}^T$, of $\bar{x}$ is a column vector.

$$\bar{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

1.2.6 <u>Matrix Sum and Difference.</u> The sum (or difference) of two $m \times n$ matrices A and B is defined as follows:

$$A \pm B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$$

EXAMPLE:

If

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -4 & 0 \\ -3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -4 & 0 \\ -3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 5 & -4 & -1 \\ 0 & -1 & 5 \end{bmatrix}$$

1.2.7 It is evident from an inspection of paragraph 1.2.6 that addition of matrices is commutative and associative; that is, $A + B = B + A$ and $A + (B + C) = (A + B) + C$.

1.2.8 Matrix Products. The product of two matrices A and B is defined only if the number of columns in A is equal to the number of rows in B. If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then their product AB is an $m \times r$ matrix defined as follows:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}$$

use as a each a scalar

$$= \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{1n} b_{n1} & a_{11} b_{12} + a_{12} b_{22} + \cdots + a_{1n} b_{n2} & \cdots & a_{11} b_{1r} + a_{12} b_{2r} + \cdots + a_{1n} b_{nr} \\ a_{21} b_{11} + a_{22} b_{21} + \cdots + a_{2n} b_{n1} & a_{21} b_{12} + a_{22} b_{22} + \cdots + a_{2n} b_{n2} & \cdots & a_{21} b_{1r} + a_{22} b_{2r} + \cdots + a_{2n} b_{nr} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} b_{11} + a_{m2} b_{21} + \cdots + a_{mn} b_{n1} & a_{m1} b_{12} + a_{m2} b_{22} + \cdots + a_{mn} b_{n2} & \cdots & a_{m1} b_{1r} + a_{m2} b_{2r} + \cdots + a_{mn} b_{nr} \end{bmatrix}$$

EXAMPLE:

If

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -4 & 0 \\ -3 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -4 & 0 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -2 & 9 \\ -15 & 2 & 4 \\ 7 & -10 & 5 \end{bmatrix}$$

and

$$AC = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -4 & 0 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -15 \\ 7 \end{bmatrix}$$

1.2.9 On inspecting paragraph 1.2.8, we see that multiplication of matrices is associative but not, in general, commutative; that is, $(AB)C = A(BC)$ but $AB \neq BA$, in general.

1.2.10 Further inspection reveals (paragraphs 1.2.6 and 1.2.8) that matrices are left and right distributive with respect to addition; that is,

$$A(B + C) = AB + AC \quad \text{and} \quad (B + C)A = BA + CA.$$

1.2.11 If 0 is the matrix (b_{ij}) such that $b_{ij} = 0$ for all i and j, then 0 is called a null or <u>zero matrix</u> and $A + 0 = A$.

1.2.12 The expression $I = (c_{ij})$ is called an <u>identity (or unit) matrix</u> when it is a square matrix whose elements are all zeros except for those on the principal diagonal (top left to lower right) which are all ones:

$$
(c_{ij}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}
$$

It can be shown that $AI = A$ and $IB = B$, where the indicated products are defined. If A is a square matrix, then $AI = IA = A$.

1.2.13 The <u>inverse</u> of matrix A is denoted A^{-1} and has the property that both $A^{-1}A$ and AA^{-1} are unit matrices.

cartesian)

1.2.14 A matrix A is <u>orthogonal</u> when $AA^T = I$, an identity matrix. If A is orthogonal, then its transpose is equal to its inverse; that is, for A orthogonal, $AA^T = I$, $A^{-1}(AA^T) = A^{-1}(I)$, $(A^{-1}A)A^T = A^{-1}$, $IA^T = A^{-1}$, and hence $A^T = A^{-1}$.

1.2.15 <u>Linear Transformations.</u> If A is an $n \times n$ matrix and $\bar{x}$ is an n^{th} order column vector, then A transforms $\bar{x}$ into $\bar{y}$ as follows:

If

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

then the linear transformation of $\bar{x}$ by A is

$$A\bar{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} \equiv \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \bar{y}$$

If A is orthogonal, then $\bar{y}$ can be transformed back into $\bar{x}$ as follows:

Since $A\bar{x} = \bar{y}$, then $A^{-1}A\bar{x} = A^{-1}\bar{y}$, and

$$\bar{x} = A^{-1}\bar{y} = A^T\bar{y} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \\ a_{12}y_1 + a_{22}y_2 + a_{32}y_3 \\ a_{13}y_1 + a_{23}y_2 + a_{33}y_3 \end{bmatrix}$$

1.3 ELEMENTARY VECTOR OPERATIONS

1.3.1 A <u>scalar</u> is a quantity that is characterized by magnitude only or by magnitude and an algebraic sign; e.g., mass, time, distance.

1.3.2 A <u>vector</u> is a quantity which has direction as well as magnitude; e.g., force, velocity, acceleration. It is represented graphically by an arrow whose length indicates magnitude (or modulus) and whose terminus indicates direction.

It follows from this definition that a vector quantity is preserved when displaced parallel to itself.

Vector quantities are denoted by a letter with an overline; e.g., $\bar{a}$. The magnitude only of a vector quantity is denoted by the letter without the overline, or by indicating the absolute value of the vector; i.e., a or $|\bar{a}|$. In matrix notation, a vector is usually columnar (paragraph 1.2.5) with each element denoting the value of the respective components of the vector in a coordinate space.

A vector $\bar{a}$ in a right-handed system of coordinate axes OXYZ has its magnitude and direction completely determined by the three component vectors $\bar{a}_x$, $\bar{a}_y$, $\bar{a}_z$ in the respective directions of the coordinate system (Fig. 1).

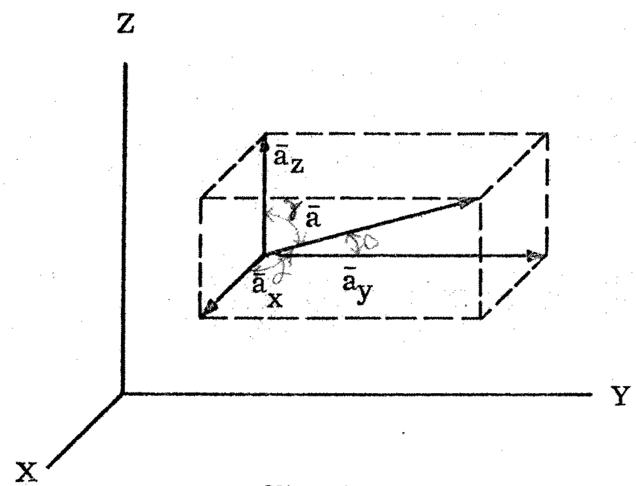


Fig. 1

In Fig. 1, it is apparent that the modulus of $\bar{a}$ is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

If α, β, and γ denote the angles between $\bar{a}$ and the three component vectors $\bar{a}_x$, $\bar{a}_y$, and $\bar{a}_z$, respectively, then

$$\cos\alpha = \frac{a_x}{a}, \quad \cos\beta = \frac{a_y}{a}, \quad \text{and} \quad \cos\gamma = \frac{a_z}{a}$$

These cosines are known as the <u>direction cosines</u> of the vector $\bar{a}$ with respect to the the coordinate system OXYZ.

1.3.3 Addition of two vectors $\bar{a}$ and $\bar{b}$ is shown in Fig. 2. Construct the origin of one of the vectors (e.g., $\bar{b}$) so that it is coincident with the terminus of the other. The resultant vector $\bar{c}$, determined by constructing a vector from the origin of $\bar{a}$ to the terminus of $\bar{b}$, is the sum of $\bar{a}$ and $\bar{b}$.

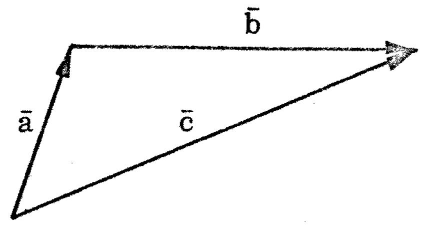


Fig. 2

Numerical values can be obtained by employing trigonometric identities. In this manner, any number of vectors can be summed.

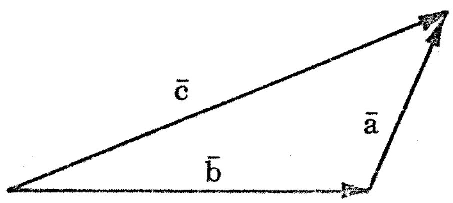


Fig. 3

If $\bar{b}$ (Fig. 3) is the vector which when added to $\bar{a}$ will give $\bar{c}$, then $\bar{a} = \bar{c} - \bar{b}$ is the difference between $\bar{c}$ and $\bar{b}$.

1.3.4 A vector $\bar{v}$ multiplied by a scalar s results in the vector $s\bar{v}$, which is s times greater in magnitude than $\bar{v}$ and in the same direction if s is positive, but opposite in direction if s is negative.

1.3.5 The vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$ (Fig. 4) are unit vectors in the direction of x, y, and z, respectively, of OXYZ. They are defined as having the property

$$|\bar{i}| = 1, \quad |\bar{j}| = 1, \quad |\bar{k}| = 1$$

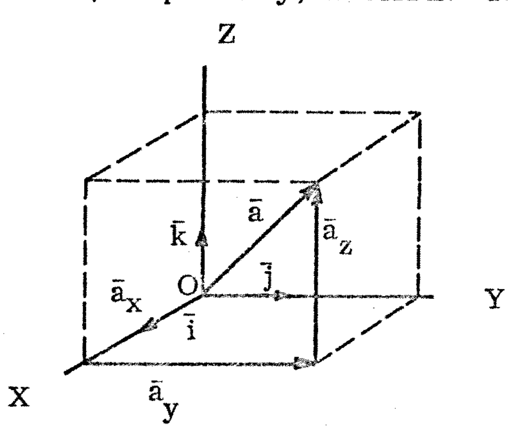


Fig. 4

Thus, if the projections (Fig. 4) of the vector $\bar{a}$ on the coordinate axes are

$$a_x, \; a_y, \; a_z$$

the vector $\bar{a}$ is said to be composed of component vectors

$$\bar{a}_x = \bar{i}a_x \, , \; \bar{a}_y = \bar{j}a_y \, , \; \bar{a}_z = \bar{k}a_z$$

Moreover, since $\bar{a} = \bar{a}_x + \bar{a}_y + \bar{a}_z$, $\bar{a}$ can also be given by $\bar{a} = \bar{i}a_x + \bar{j}a_y + \bar{k}a_z$.

1.3.6 Scalar (Dot) Product of Two Vectors. The scalar or dot product of two vectors is equal to the product of their magnitudes, multiplied by the cosine of their included angle (see Fig. 5).

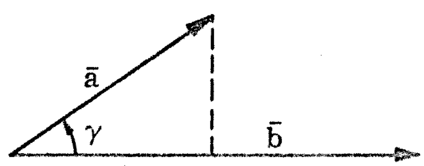


Fig. 5

Symbolically,

$$\bar{a} \cdot \bar{b} = |\bar{a}| \, |\bar{b}| \cos \gamma$$

It follows that scalar products between unit vectors give the special relationships

$$\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1 \text{ and } \bar{i} \cdot \bar{j} = \bar{i} \cdot \bar{k} = \bar{j} \cdot \bar{k} = 0$$

Consequently, for

$$\bar{a} = \bar{i}a_x + \bar{j}a_y + \bar{k}a_z \text{ and } \bar{b} = \bar{i}b_x + \bar{j}b_y + \bar{k}b_z$$

$$\begin{aligned}
\bar{a} \cdot \bar{b} &= (\bar{i}a_x + \bar{j}a_y + \bar{k}a_z) \cdot (\bar{i}b_x + \bar{j}b_y + \bar{k}b_z) \\
&= (a_x b_x)\,\bar{i} \cdot \bar{i} + (a_x b_y)\,\bar{i} \cdot \bar{j} + (a_x b_z)\,\bar{i} \cdot \bar{k} + (a_y b_x)\,\bar{j} \cdot \bar{i} \\
&\quad + (a_y b_y)\,\bar{j} \cdot \bar{j} + (a_y b_z)\,\bar{j} \cdot \bar{k} + (a_z b_x)\,\bar{k} \cdot \bar{i} + (a_z b_y)\,\bar{k} \cdot \bar{j} \\
&\quad + (a_z b_z)\,\bar{k} \cdot \bar{k} \\
&= a_x b_x + a_y b_y + a_z b_z
\end{aligned}$$

The scalar square of a vector gives

$$\bar{a} \cdot \bar{a} = |\bar{a}|^2 \cos(0) = |\bar{a}|^2$$

1.3.7 Vector (Cross) Product of Two Vectors. The vector or cross product $\bar{a} \times \bar{b}$ of two vectors $\bar{a}$ and $\bar{b}$ (Fig. 6) is equal to the vector whose magnitude is the product of their magnitudes multiplied by the sine of their included angle. The cross product is perpendicular to the plane of the two vectors such that a rotation of vector $\bar{a}$ about the cross product into vector $\bar{b}$ would effect a right-hand screw to advance in the direction of the cross product. Symbolically,

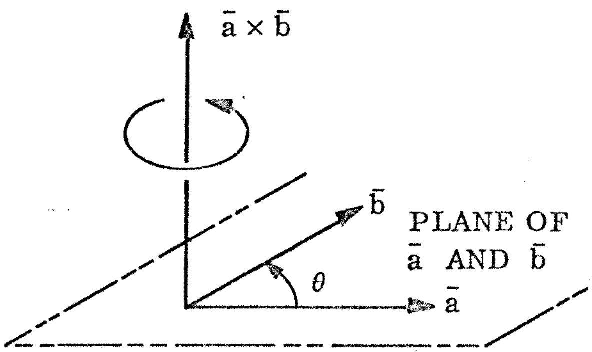


Fig. 6

$$|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}| \sin\theta$$

Since the rotation that carries $\bar{b}$ into $\bar{a}$ is opposite to that which carries $\bar{a}$ into $\bar{b}$, then

$$\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$$

If $\theta = 0$ or $180°$, then

$$|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}| \sin\theta = 0$$

Also,

$$|\bar{a} \times \bar{a}| = |\bar{a}|^2 \sin\theta = 0$$

Cross products between unit vectors give

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0$$

$$\bar{i} \times \bar{j} = \bar{k} = -\bar{j} \times \bar{i}$$

$$\bar{j} \times \bar{k} = \bar{i} = -\bar{k} \times \bar{j}$$

$$\bar{k} \times \bar{i} = \bar{j} = -\bar{i} \times \bar{k}$$

Consequently, for

$$\bar{a} = \bar{i}a_x + \bar{j}a_y + \bar{k}a_z \quad \text{and} \quad \bar{b} = \bar{i}b_x + \bar{j}b_y + \bar{k}b_z$$

$$\begin{aligned}
\bar{a} \times \bar{b} &= (\bar{i}a_x + \bar{j}a_y + \bar{k}a_z) \times (\bar{i}b_x + \bar{j}b_y + \bar{k}b_z) \\
&= (a_x b_x)\,\bar{i}\times\bar{i} + (a_x b_y)\,\bar{i}\times\bar{j} + (a_x b_z)\,\bar{i}\times\bar{k} + (a_y b_x)\,\bar{j}\times\bar{i} + (a_y b_y)\,\bar{j}\times\bar{j} \\
&\quad + (a_y b_z)\,\bar{j}\times\bar{k} + (a_z b_x)\,\bar{k}\times\bar{i} + (a_z b_y)\,\bar{k}\times\bar{j} + (a_z b_z)\,\bar{k}\times\bar{k} \\
&= (a_x b_y)\bar{k} - (a_x b_z)\bar{j} - (a_y b_x)\bar{k} + (a_y b_z)\bar{i} + (a_z b_x)\bar{j} - (a_z b_y)\bar{i} \\
&= (a_y b_z - a_z b_y)\bar{i} + (a_z b_x - a_x b_z)\bar{j} + (a_x b_y - a_y b_x)\bar{k}
\end{aligned}$$

which can be written in determinant form as

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

1.3.8 Radius Vector. Consider a point (x , y , z) in the right-hand system OXYZ (Fig. 7). The vector determined by the origin O and the point (x , y , z), where (x , y , z) is the terminus, is called the radius vector $\bar{r}$ of point (x , y , z), and

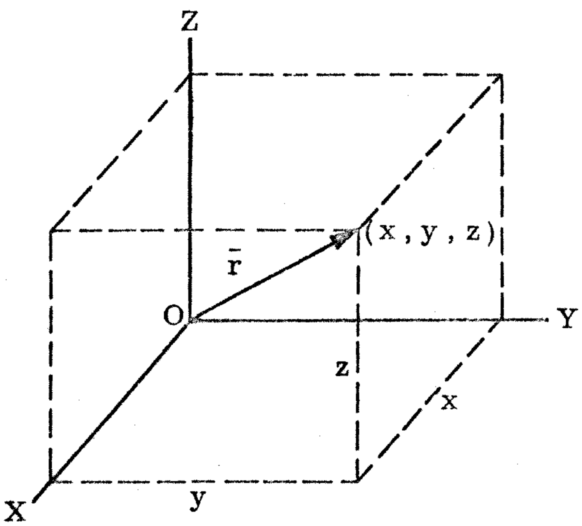


Fig. 7

$$\bar{r} = \bar{i}x + \bar{j}y + \bar{k}z$$

If $\bar{r}$ indicates the motion of a point with respect to time, then, in function notation,

$$\bar{r} = \bar{r}(t)$$

and $x = x(t)$, $y = y(t)$, $z = z(t)$

1.3.9 Vector Derivatives. Consider a vector $\bar{a}$ which changes in magnitude and direction correspondingly with a change in a scalar variable t ; that is,

$$\bar{a} = \bar{f}(t)$$

If t increases infinitesimally by Δt , then $\bar{a}$ changes infinitesimally by $\Delta\bar{a}$ (Fig. 8). Symbolically,

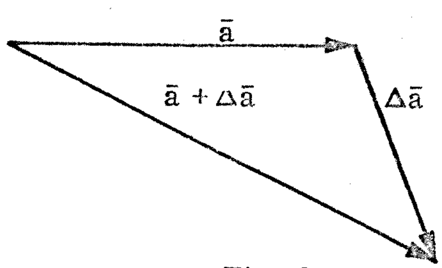


Fig. 8

$$\bar{a} + \Delta\bar{a} = \bar{f}(t + \Delta t) .$$

Hence, the vector differential of $\bar{a}$ is

$$\Delta\bar{a} = \bar{f}(t + \Delta t) - \bar{f}(t)$$

and the instantaneous rate of change of $\bar{a}$ with respect to t as Δt approaches zero as a limit is

$$\frac{d\bar{a}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \bar{a}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{f}(t + \Delta t) - \bar{f}(t)}{\Delta t}$$

which is the vector derivative of $\bar{a}$ with respect to t.

The second derivative of $\bar{a}$ is obtained in the same way, as in

$$\frac{d^2\bar{a}}{dt^2} = \lim_{\Delta t \to 0} \frac{\Delta^2 \bar{a}}{\Delta t^2} = \lim_{\Delta t \to 0} \frac{\Delta \left[\frac{\bar{f}(t + \Delta t) - \bar{f}(t)}{\Delta t}\right]}{\Delta t}$$

Higher order derivatives of $\bar{a}$ can be similarly obtained. If k is a scalar variable which is also a function of t, that is,

$$k = F(t) ,$$

then the differential of the product $\bar{u}$ of the scalar function k and the vector function $\bar{a} = \bar{f}(t)$ is

$$\begin{aligned} \Delta(k\bar{a}) = \Delta\bar{u} &= F(t + \Delta t)\,\bar{f}(t + \Delta t) - F(t)\,\bar{f}(t) \\ &= F(t + \Delta t)\,\bar{f}(t + \Delta t) - F(t + \Delta t)\,\bar{f}(t) + F(t + \Delta t)\,\bar{f}(t) - F(t)\,\bar{f}(t) \end{aligned}$$

and the vector derivative of $\bar{u}$ with respect to t is

$$\begin{aligned} \frac{d\bar{u}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \bar{u}}{\Delta t} &= \lim_{\Delta t \to 0} F(t + \Delta t) \left[\frac{\bar{f}(t + \Delta t) - \bar{f}(t)}{\Delta t}\right] \\ &\quad + \lim_{\Delta t \to 0} \left[\frac{F(t + \Delta t) - F(t)}{\Delta t}\right] \bar{f}(t) \end{aligned}$$

$$= k\frac{d\bar{a}}{dt} + \frac{dk}{dt}\bar{a} \equiv \frac{d(k\bar{a})}{dt}$$

Consider the two vectors $\bar{a}$ and $\bar{b}$ which are functions of a scalar t. The scalar product of $\bar{a}$ and $\bar{b}$ is

$$\bar{a} \cdot \bar{b} = a_x b_x + a_y b_y + a_z b_z ,$$

and the vector derivative of $\bar{a} \cdot \bar{b}$ is

$$\frac{d(\bar{a} \cdot \bar{b})}{dt} = a_x \frac{db_x}{dt} + b_x \frac{da_x}{dt} + a_y \frac{db_y}{dt} + b_y \frac{da_y}{dt} + a_z \frac{db_z}{dt} + b_z \frac{da_z}{dt}$$

$$= \bar{a} \cdot \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \cdot \bar{b}$$

The vector product of $\bar{a}$ and $\bar{b}$ is

$$\bar{a} \times \bar{b} = (a_y b_z - a_z b_y)\bar{i} + (a_z b_x - a_x b_z)\bar{j} + (a_x b_y - a_y b_x)\bar{k}$$

and the vector derivative of $\bar{a} \times \bar{b}$, where $\bar{i}$, $\bar{j}$, and $\bar{k}$ are constant, is

$$\frac{d(\bar{a} \times \bar{b})}{dt} = \frac{d}{dt}\left[(a_y b_z - a_z b_y)\bar{i} + (a_z b_x - a_x b_z)\bar{j} + (a_x b_y - a_y b_x)\bar{k}\right]$$

$$= (a_y \dot{b}_z - a_z \dot{b}_y)\bar{i} + (\dot{a}_y b_z - \dot{a}_z b_y)\bar{i} + (a_z \dot{b}_x - a_x \dot{b}_z)\bar{j}$$

$$+ (\dot{a}_z b_x - \dot{a}_x b_z)\bar{j} + (a_x \dot{b}_y - a_y \dot{b}_x)\bar{k} + (\dot{a}_x b_y - \dot{a}_y b_x)\bar{k}$$

$$= \bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \times \bar{b}$$

1.4 EULER ANGLES AND DIRECTION COSINES

1.4.1 Consider a point P relative to the right-handed coordinate system OXYZ (Fig. 9). The angles α, β, and γ between the ray OP and the X, Y, and Z axes, respectively, are <u>direction angles</u> of OP. The cosines of these angles

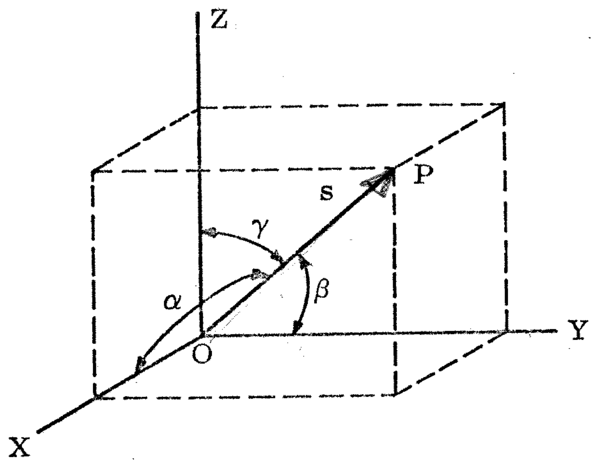


Fig. 9

$$\cos\alpha = \frac{x}{s}, \quad \cos\beta = \frac{y}{s} \quad \text{and} \quad \cos\gamma = \frac{z}{s}$$

where

$$s = \sqrt{x^2 + y^2 + z^2}$$

are called the direction cosines of the ray OP.

Now consider two right-handed coordinate systems OXYZ and OX'Y'Z' which have a common origin (Fig. 10). If the angles between respective coordinate axes are given by

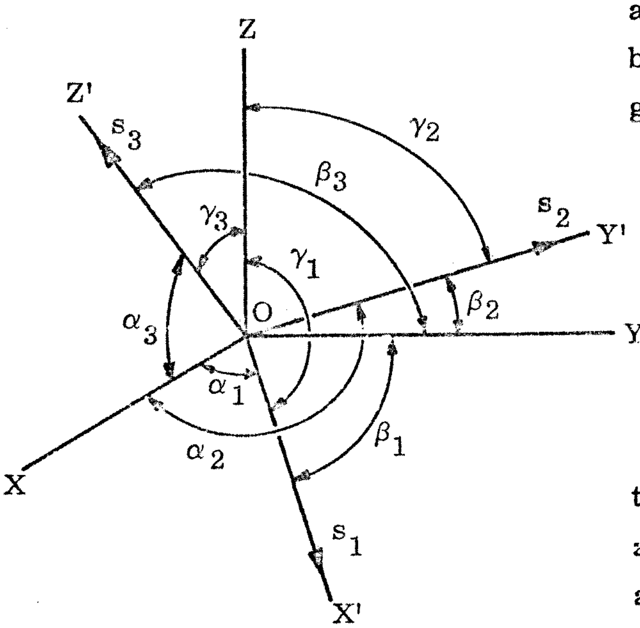


Fig. 10

	OX	OY	OZ
OX'	α_1	β_1	γ_1
OY'	α_2	β_2	γ_2
OZ'	α_3	β_3	γ_3

then the direction cosines of the OX', OY', and OZ' axes relative to the OX, OY, and OZ axes are

$$C_{x'x} = \cos\ \alpha_1 = \frac{x_1}{s_1},\ C_{x'y} = \cos\ \beta_1 = \frac{y_1}{s_1},\ C_{x'z} = \cos\ \gamma_1 = \frac{z_1}{s_1}$$

$$C_{y'x} = \cos\ \alpha_2 = \frac{x_2}{s_2},\ C_{y'y} = \cos\ \beta_2 = \frac{y_2}{s_2},\ C_{y'z} = \cos\ \gamma_2 = \frac{z_2}{s_2}$$

$$C_{z'x} = \cos\ \alpha_3 = \frac{x_3}{s_3},\ C_{z'y} = \cos\ \beta_3 = \frac{y_3}{s_3},\ C_{z'z} = \cos\ \gamma_3 = \frac{z_3}{s_3}$$

where

$$s_1 = \sqrt{x_1^2 + y_1^2 + z_1^2},\ s_2 = \sqrt{x_2^2 + y_2^2 + z_2^2},\ s_3 = \sqrt{x_3^2 + y_3^2 + z_3^2}$$

and the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are on the OX', OY', and OZ' axes, respectively.

Using vector notation, the point (x_1, y_1, z_1) is the terminus of the vector

$$\bar{s}_1 = \bar{x}_1 + \bar{y}_1 + \bar{z}_1$$

where

$$\bar{x}_1 = \bar{i}\ x_1,\ \bar{y}_1 = \bar{j}\ y_1,\ \text{and}\ \bar{z}_1 = \bar{k}\ z_1$$

Since $\bar{s}_1$ lies along the OX' axis, the unit vector $\bar{i}'$ along this axis is given by

or $\hat{\imath}$ $$\bar{i}' = \frac{\bar{s}_1}{s_1} = \frac{\bar{x}_1 + \bar{y}_1 + \bar{z}_1}{s_1}$$

$$\hat{\imath} = \bar{i}\,\frac{x_1}{s_1} + \bar{j}\,\frac{y_1}{s_1} + \bar{k}\,\frac{z_1}{s_1}$$

$$\hat{\imath} = \bar{i}\ \cos\ \alpha_1 + \bar{j}\ \cos\ \beta_1 + \bar{k}\ \cos\ \gamma_1$$

The above process, then, produces the three unit vectors

$$\bar{i}' = \bar{i}\cos\alpha_1 + \bar{j}\cos\beta_1 + \bar{k}\cos\gamma_1$$

$$\bar{j}' = \bar{i}\cos\alpha_2 + \bar{j}\cos\beta_2 + \bar{k}\cos\gamma_2$$

$$\bar{k}' = \bar{i}\cos\alpha_3 + \bar{j}\cos\beta_3 + \bar{k}\cos\gamma_3$$

which give the following table of direction cosines:

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{i}'$	$C_{x'x}$	$C_{x'y}$	$C_{x'z}$
$\bar{j}'$	$C_{y'x}$	$C_{y'y}$	$C_{y'z}$
$\bar{k}'$	$C_{z'x}$	$C_{z'x}$	$C_{z'z}$

The nine direction cosines can be expressed as the matrix

$$A = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix}$$

where the prime superscripts have been omitted for brevity. The transpose of matrix A is then

$$A^T = \begin{bmatrix} C_{xx} & C_{yx} & C_{zx} \\ C_{xy} & C_{yy} & C_{zy} \\ C_{xz} & C_{yz} & C_{zz} \end{bmatrix}$$

Consider the matrix product

$$AA^T = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix} \begin{bmatrix} C_{xx} & C_{yx} & C_{zx} \\ C_{xy} & C_{yy} & C_{zy} \\ C_{xz} & C_{yz} & C_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \left(C_{xx}^2 + C_{xy}^2 + C_{xz}^2\right) & \left(C_{xx}C_{yx} + C_{xy}C_{yy} + C_{xz}C_{yz}\right) & \left(C_{xx}C_{zx} + C_{xy}C_{zy} + C_{xz}C_{zz}\right) \\ \left(C_{yx}C_{xx} + C_{yy}C_{xy} + C_{yz}C_{xz}\right) & \left(C_{yx}^2 + C_{yy}^2 + C_{yz}^2\right) & \left(C_{yx}C_{zx} + C_{yy}C_{zy} + C_{yz}C_{zz}\right) \\ \left(C_{zzCxx} + C_{zy}C_{xy} + C_{zz}C_{xz}\right) & \left(C_{zx}C_{yx} + C_{zy}C_{yy} + C_{zz}C_{yz}\right) & \left(C_{zx}^2 + C_{zy}^2 + C_{zz}^2\right) \end{bmatrix}$$

Each element of the above matrix can be shown to be equivalent to some scalar product of two of the unit vectors $\bar{i}'$, $\bar{j}'$, $\bar{k}'$. For example,

$$\bar{i}' \cdot \bar{i}' = (\bar{i}\, C_{xx} + \bar{j}\, C_{xy} + \bar{k}\, C_{xz}) \cdot (\bar{i}\, C_{xx} + \bar{j}\, C_{xy} + \bar{k}\, C_{xz})$$

$$= C_{xx}^2 + C_{xy}^2 + C_{xz}^2$$

By substitution, there results the matrix of scalar products

$$\begin{bmatrix} \bar{i}' \cdot \bar{i}' & \bar{i}' \cdot \bar{j}' & \bar{i}' \cdot \bar{k}' \\ \bar{j}' \cdot \bar{i}' & \bar{j}' \cdot \bar{j}' & \bar{j}' \cdot \bar{k}' \\ \bar{k}' \cdot \bar{i}' & \bar{k}' \cdot \bar{j}' & \bar{k}' \cdot \bar{k}' \end{bmatrix}$$

Hence, the product

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the identity matrix I (Section 1. 2. 12).

Therefore, the matrix A of direction cosines is orthogonal (Section 1. 2. 14), and its transpose is equal to its inverse.

Suppose now that a vector $\bar{p}$ is given by

$$\bar{p} = \bar{i}'x' + \bar{j}'y' + \bar{k}'z'$$

Then, substituting the preceding relationships for $\bar{i}'$, $\bar{j}'$, and $\bar{k}'$ we obtain

$$\bar{p} = \bar{i}x'C_{xx} + \bar{j}x'C_{xy} + \bar{k}x'C_{xz} + \bar{i}y'C_{yx} + \bar{j}y'C_{yy} + \bar{k}y'C_{yz} + \bar{i}z'C_{zx} + \bar{j}z'C_{zy} + \bar{k}z'C_{zz}$$

again omitting the superscripts from the C's for brevity. Regrouping in terms of $\bar{i}$, $\bar{j}$, and $\bar{k}$, we obtain

$$\bar{p} = \bar{i}(x'C_{xx} + y'C_{yx} + z'C_{zx}) + \bar{j}(x'C_{xy} + y'C_{yy} + z'C_{zy}) + \bar{k}(x'C_{yz} + y'C_{yz} + z'C_{zz})$$

The quantities in parentheses are thus precisely the components of the vector $\bar{p}$ expressed in the OXYZ coordinate system. If we call these coordinates x , y , and z , then

$$x = x'C_{xx} + y'C_{yx} + z'C_{zx}$$

$$y = x'C_{xy} + y'C_{yy} + z'C_{zy}$$

$$z = x'C_{yz} + y'C_{yz} + z'C_{zz}$$

These relationships can be written in vector-matrix notation as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{yx} & C_{zx} \\ C_{xy} & C_{yy} & C_{zy} \\ C_{xz} & C_{yz} & C_{zz} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

The matrix of direction cosines is said to be the matrix which <u>transforms</u> the OX'Y'Z' coordinate system into the OXYZ coordinate system.

Since this matrix is orthogonal, we can write

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{yx} & C_{zx} \\ C_{xy} & C_{yy} & C_{zy} \\ C_{xz} & C_{yz} & C_{zz} \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C_{xx}x + C_{xy}y + C_{xz}z \\ C_{yx}x + C_{yy}y + C_{yz}z \\ C_{zx}x + C_{zy}y + C_{zz}z \end{bmatrix}$$

which is the transformation from OXYZ back into OX'Y'Z'.

Another method of relating the two coordinate frames OXYZ and OX'Y'Z' involves the use of Euler (Eulerian) angles. These angles are obtained by determining the successive angles of rotation through which one frame must be rotated in order to make it coincident with the other frame. For example, OXYZ can be brought into coincidence with OX'Y'Z' by:

NOTE

A different perspective is used in Figs. 11 through 13 from that used in Fig. 10.

(1) A rotation through an angle ϕ about OZ to make OY coincident with OV, which is the line of intersection of the planes OXY and OX'Y' (Fig. 11);

(2) A rotation through an angle θ about OV, the new position of OY, to make OZ coincident with OZ' (Fig. 12); and

(3) A rotation through an angle ψ about OZ' (the new position of OZ) to make OV (the new position of OY) coincident with OY' (Fig. 13).

During the three rotations, OX went in turn to OW, OW', then OX'.

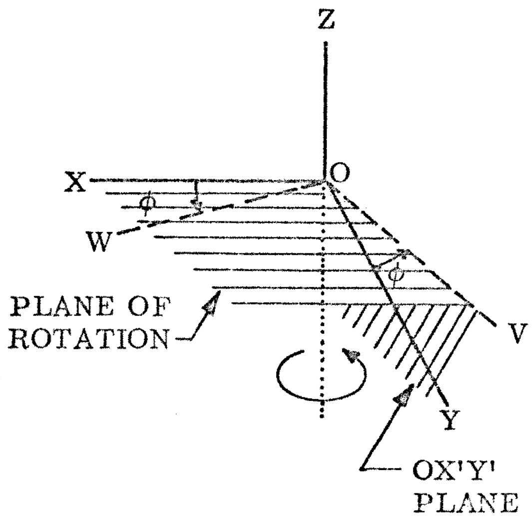


Fig. 11

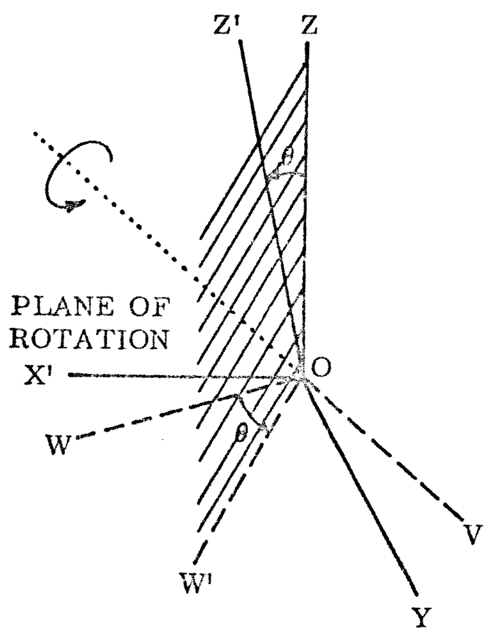


Fig. 12

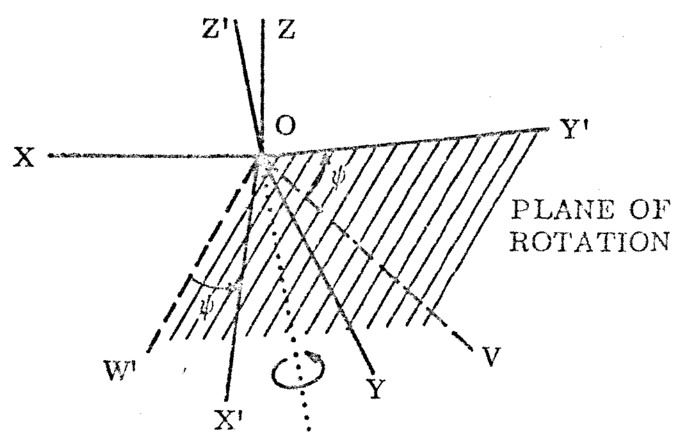


Fig. 13

The angles ϕ , θ , and ψ are the Euler angles. The relative orientation of two coordinate systems in terms of Euler angles is developed in the next section.

1.4.2 Roll, Pitch, and Yaw. Consider a single rotation through angle ϕ of axis OY' from axis OY (Fig. 14). The direction cosines relating the two frames OXYZ and OX'Y'Z' would then be

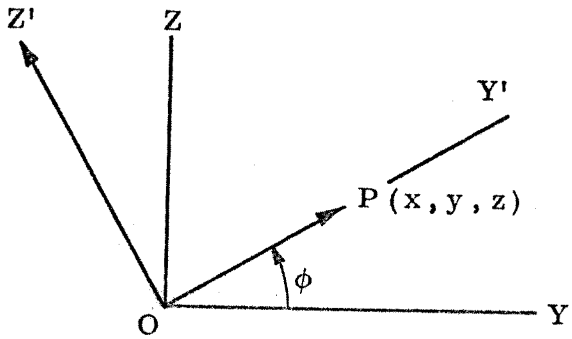
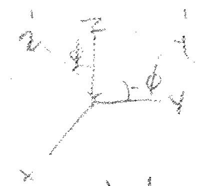


Fig. 14



about x +

$$
\begin{array}{c|c}
 & \begin{array}{ccc} OX & OY & OZ \end{array} \\
\hline
\begin{array}{c} OX' \\ OY' \\ OZ' \end{array} & \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix}
\end{array}
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \cos\left(\frac{\pi}{2} - \phi\right) \\ 0 & \cos\left(\frac{\pi}{2} + \phi\right) & \cos\phi \end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}
$$

The point P can be referenced to the new frame through the following relations of coordinates x' , y' , and z' :

$$x' = xC_{xx} + yC_{xy} + zC_{xz} = x$$

$$y' = xC_{yx} + yC_{yy} + zC_{yz} = y\cos\phi + z\sin\phi$$

$$z' = xC_{zx} + yC_{zy} + zC_{zz} = -y\sin\phi + z\cos\phi$$

These equations may be expressed as the matrix equations

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

is commonly known as the roll matrix; that is, a rotation of the YZ plane about the X axis is known as a roll.

1.4.3 Generating the X'Y'Z' frame by a rotation of the XZ plane through an angle θ about the Y axis (Fig. 15) results in the direction cosines

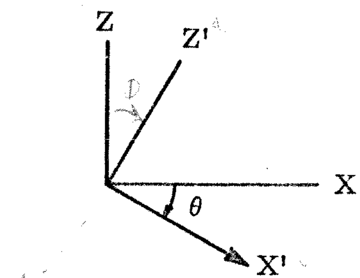


Fig. 15

$$\begin{array}{c|c} & \begin{matrix} OX & OY & OZ \end{matrix} \\ \hline \begin{matrix} OX' \\ OY' \\ OZ' \end{matrix} & \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix} \end{array} = \begin{bmatrix} \cos\theta & 0 & \cos(\frac{\pi}{2}+\theta) \\ 0 & 1 & 0 \\ \cos(\frac{\pi}{2}-\theta) & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

which is commonly known as the pitch matrix, and also results in the referencing of point P to the new frame through the following relations of coordinates x' , y' , and z' :

$$x' = xC_{xx} + yC_{xy} + zC_{xz} = x\cos\theta - z\sin\theta$$

$$y' = xC_{yx} + yC_{yy} + zC_{yz} = y$$

$$z' = xC_{zx} + yC_{zy} + zC_{zz} = x\sin\theta + z\cos\theta$$

pitch

These equations may be expressed as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

1.4.4 Generating the $X'Y'Z'$ frame by rotation of the XY plane through an angle ψ about the Z axis (Fig. 16) results in the direction cosines

yaw

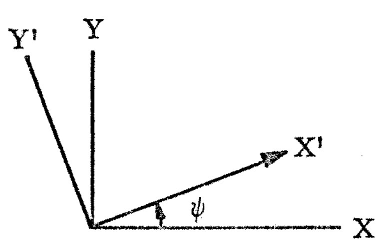


Fig. 16

$$\begin{array}{c|ccc} & OX & OY & OZ \\ \hline OX' & & & \\ OY' & & \left[\begin{matrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{matrix}\right] & \\ OZ' & & & \end{array} = \begin{bmatrix} \cos\psi & \cos(\frac{\pi}{2} - \psi) & 0 \\ \cos(\frac{\pi}{2} + \psi) & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is commonly known as the yaw matrix, and also results in the referencing of point P to the new frame through the following relations of coordinates x' , y' , and z' :

$$x' = xC_{xx} + yC_{xy} + zC_{xz} = x \cos \psi + y \sin \psi$$

$$y' = xC_{yx} + yC_{yy} + zC_{yz} = -x \sin \psi + y \cos \psi$$

$$z' = xC_{zx} + yC_{zy} + zC_{zz} = z$$

These equations may be expressed as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

1.4.5 Suppose we wish to accomplish a roll, followed by a pitch, followed by a yaw. If A is the roll matrix, B the pitch matrix, and C the yaw matrix, then the process would be

$$\text{Roll} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{Pitch} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = B \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \qquad \text{Yaw} \begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = C \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

By substitution, we obtain

$$\begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = CBA \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Thus, it would be possible to first perform the matrix multiplication CBA (which is not, in general, commutative), thereby resolving the roll, pitch, and yaw to the single matrix M , as in

$$\begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = CBA \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

1.4.6 A roll about the X axis, followed by a pitch about the resulting Y' axis, followed by a yaw about the resulting Z' axis may be accomplished as follows:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Yaw}} \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{\text{Pitch}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{\text{Roll}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta\sin\phi & -\sin\theta\cos\phi \\ 0 & \cos\phi & \sin\phi \\ \sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\psi = \beta$
$\theta = \alpha$

Pitch-YAW

$$\begin{bmatrix} C\alpha C\beta & S\alpha & -C\alpha S\beta \\ -S\alpha C\beta & +C\alpha & +S\alpha S\beta \\ S\beta & 0 & C\beta \end{bmatrix}$$

$$
= \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi + \sin\psi\cos\phi & -\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ -\sin\psi\cos\theta & -\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi + \cos\psi\sin\phi \\ \sin\theta & -\cos\theta\sin\varphi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

In what follows, we show that the nine elements of the transformation matrix in terms of the Euler angles ϕ , ψ , and θ are precisely the direction cosines relating the OXYZ and OX'Y'Z' frames.

For simplicity, let the above transformation matrix be represented by the matrix

$$
D = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix}
$$

Let us define three vectors $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ with the following components in the OXYZ coordinate system:

$$\bar{p}_1 = \bar{i}D_{xx} + \bar{j}D_{xy} + \bar{k}D_{xz}$$

$$\bar{p}_2 = \bar{i}D_{yx} + \bar{j}D_{yy} + \bar{k}D_{yz}$$

$$\bar{p}_3 = \bar{i}D_{zx} + \bar{j}D_{zy} + \bar{k}D_{zz}$$

We now find the components of $\bar{p}_1$ in the OX'Y'Z' coordinate system by applying the transformation D to $\bar{p}_1$

$$\bar{p}_1 = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{bmatrix} D_{xx} \\ D_{xy} \\ D_{xz} \end{bmatrix}$$

$$= (D_{xx}^2 + D_{xy}^2 + D_{xz}^2)\,\bar{i}' + (D_{yx}D_{xx} + D_{yy}D_{xy} + D_{yz}D_{xz})\,\bar{j}' + (D_{zx}D_{xx} + D_{zy}D_{xy} + D_{zz}D_{xz})\,\bar{k}'$$

By actual substitution of the trigonometric identities for the elements of matrix D (which is left as an exercise for the reader), we obtain

$$\bar{p}_1 = \bar{i}' + 0\bar{j}' + 0\bar{k}' = \bar{i}'$$

Thus, $\bar{p}_1$ is the unit vector along the X' axis of the OX'Y'Z' coordinate system. Similarly, we can show that $\bar{p}_2 = \bar{j}'$ and $\bar{p}_3 = \bar{k}'$.

The angles between the OX'Y'Z' and OXYZ axes were listed previously and are repeated here for convenience.

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{i}'$	α_1	β_1	γ_1
$\bar{j}'$	α_2	β_2	γ_2
$\bar{k}'$	α_3	β_3	γ_3

Now

$$\bar{i}' \cdot \bar{i} = \cos \alpha_1 \equiv C_{xx}$$

But

$$\bar{i}' \cdot \bar{i} = \bar{p}_1 \cdot \bar{i} = (\bar{i}D_{xx} + \bar{j}D_{xy} + \bar{k}D_{xz}) \cdot (\bar{i} + 0\bar{j} + 0\bar{k}) = D_{xx}$$

Therefore,

$$D_{xx} = C_{xx}$$

Similarly,

$$\bar{i}' \cdot \bar{j} = \cos \beta_1 = C_{xy}$$

but

$$\bar{i}' \cdot \bar{j} = \bar{p}_1 \cdot \bar{j} = (\bar{i}D_{xx} + \bar{j}D_{xy} + \bar{k}D_{xz}) \cdot (0\bar{i} + \bar{j} + 0\bar{k}) = D_{xy}$$

and, therefore,

$$D_{xy} = C_{xy}$$

By similar means, we can show that

$$\begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix}$$

Thus, the elements of matrix D are the direction cosines relating the OX'Y'Z' and OXYZ coordinate systems, and we have shown how to compute the direction cosines from the Euler angles ϕ, θ, and ψ.

We can now simplify our notation in expressing the roll, pitch, yaw transformation to

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C_{xx}x + C_{xy}y + C_{xz}z \\ C_{yx}x + C_{yy}y + C_{yz}z \\ C_{zx}x + C_{zy}y + C_{zz}z \end{bmatrix}$$

Since matrix multiplication is not, in general, commutative, it is essential that the transformations be accomplished in the order implied. The example above exhibits first a roll (by showing it as the first multiplier of the column vector), then a pitch, and finally a yaw. The number and order of rotations is determined by the nature of the physical problem.